

# A NEW CLASS OF (3+1)-DIMENSIONAL INTEGRABLE SYSTEMS RELATED TO CONTACT GEOMETRY

A. SERGYEYEV

We introduce a new kind of nonlinear Lax-type representation, a *contact Lax pair*, related to contact geometry. The compatibility conditions for the contact Lax pairs yield a broad new class of (3+1)-dimensional integrable systems, thus demonstrating that such systems are considerably less exceptional than it was generally believed. Integrability of the systems from this new class is established as we construct for them, in addition to contact Lax pairs, *linear* Lax representations. The latter make it possible to solve the systems in question using powerful techniques like the inverse scattering transform or the dressing method.

To illustrate our results, we present *inter alia* a new (3+1)-dimensional integrable system with an arbitrarily large finite number of components. In the simplest special case this system yields a (3+1)-dimensional integrable generalization of the dispersionless Kadomtsev–Petviashvili equation.

## 1. INTRODUCTION

Since the discovery of the inverse scattering transform [1] it became clear that integrable nonlinear partial differential systems are of tremendous significance both in physics and in pure and applied mathematics, see e.g. [2]–[8]. The integrable systems in four independent variables are, of course, particularly important, as our spacetime is four-dimensional. However, it was generally believed that such systems are, unlike their (2+1)-dimensional counterparts, very rare. Finding a systematic approach to the construction of integrable systems in (3+1) dimensions has been one of the most important open problems in modern theory of integrable systems, see e.g. [2, 5].

Below we present such a systematic approach based on a new kind of nonlinear Lax-type representation, a *contact Lax pair*. In Section 2 we introduce the contact Lax pairs, reveal their relation to contact geometry, and show that their compatibility conditions yield (3+1)-dimensional dispersionless systems. Integrability of the latter is established in Section 3, where we construct for them *linear* nonisospectral Lax pairs. Section 4 touches upon the structure of systems under study. In Section 5 we give examples of integrable systems arising from our construction, Section 6 addresses the reduction of our systems to (2+1) dimensions, and in Section 7 we discuss open problems.

## 2. CONTACT LAX PAIRS AND ASSOCIATED (3+1)-DIMENSIONAL INTEGRABLE SYSTEMS

The *contact Lax pair* is an overdetermined system of the form

$$\psi_y = \psi_z f(\mathbf{u}, p), \quad \psi_t = \psi_z g(\mathbf{u}, p) \quad (1)$$

for the scalar pseudopotential  $\psi = \psi(x, y, z, t)$ , where  $x, y, z, t$  are independent variables,  $p = \psi_x / \psi_z$ ,  $\mathbf{u} = \mathbf{u}(x, y, z, t)$  is an  $N$ -component vector of dependent variables,  $\mathbf{u} = (u^1, \dots, u^N)^T$ . As usual, the subscripts  $x, y, z, t, p, u^i$  stand for the partial derivatives in the respective variables, and the superscript  $T$  indicates the transposed matrix. All functions are assumed to be sufficiently smooth for all computations to make sense.

To see which nonlinear partial differential systems for  $\mathbf{u}$  arise from the compatibility condition for (1),  $(\psi_y)_t = (\psi_t)_y$ , note that the latter is readily checked to be equivalent to a zero-curvature-type equation

$$f_t - g_y + \{f, g\}_L = 0. \quad (2)$$

Here and below, in contrast with (1),  $p$  is treated as an independent variable, so  $f$  and  $g$  are assumed to depend on  $x, y, z, t$  through  $\mathbf{u}$  only. The bracket  $\{, \}_L$  is the *contact* (or *Lagrange*) *bracket*, see e.g. [9]:

$$\{h_1, h_2\}_L \stackrel{\text{df}}{=} (h_1)_p (h_2)_x - (h_2)_p (h_1)_x - p((h_1)_p (h_2)_z - (h_2)_p (h_1)_z) + h_1 (h_2)_z - h_2 (h_1)_z.$$

This bracket is skew-symmetric and satisfies the Jacobi identity but it does not obey the Leibniz rule, so an algebra of functions closed under the contact bracket is a *generalized Poisson algebra*, cf. e.g. [10].

Equation (2) is the very reason we refer to (1) as to the *contact Lax pair*: (1) is nothing but a pair of *contact* Hamilton–Jacobi equations whose compatibility condition is given by (2).

By assumption,  $f(\mathbf{u}, p)$  and  $g(\mathbf{u}, p)$  now depend on  $x, y, z, t$  through  $\mathbf{u}$  only, so we can spell out (2) as

$$\sum_{i=1}^N (f_{u^i} u_t^i - g_{u^i} u_y^i + (f_p g_{u^i} - g_p f_{u^i}) u_x^i + ((f - p f_p) g_{u^i} - (g - p g_p) f_{u^i}) u_z^i) = 0.$$

Hence the compatibility condition (2) for (1) has, upon an appropriate splitting in  $p$ , the general form

$$A_0(\mathbf{u})\mathbf{u}_t + A_1(\mathbf{u})\mathbf{u}_x + A_2(\mathbf{u})\mathbf{u}_y + A_3(\mathbf{u})\mathbf{u}_z = 0, \quad (3)$$

where  $A_i$  are  $M \times N$  matrices (and usually  $M \geq N$ ), cf. e.g. [11, 12] for the (2+1)-dimensional case.

### 3. LINEAR LAX PAIRS AND INTEGRABILITY

Our goal now is to show that systems (3) arising from the compatibility condition (2) of (1) are indeed integrable. The *de facto standard* definition of integrability<sup>1</sup> for a nonlinear partial differential system is essentially as follows: there is an overdetermined *linear* partial differential system (a Lax-type representation) compatible by virtue of the nonlinear system in question, cf. e.g. [2, 5, 7, 13, 14] for further details.

To construct such a linear system for (3), consider the contact vector fields (cf. e.g. [9])

$$X_h = h_p \partial_x + (ph_z - h_x) \partial_p + (h - ph_p) \partial_z, \quad (4)$$

associated to functions  $h$ , and recall that the map  $h \mapsto X_h$  is a Lie algebra homomorphism, so  $X_{\{h_1, h_2\}_L} = [X_{h_1}, X_{h_2}]$ , where  $[\cdot, \cdot]$  stands for the usual commutator (the Lie bracket) of vector fields. Therefore, (2) implies that  $[\partial_y - X_f, \partial_t - X_g] = 0$ , and this immediately yields a linear system compatible by virtue of (3):

**Theorem 1.** *If (3) has a contact Lax pair (1), then it also has a linear nonisospectral Lax pair*

$$\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi) \quad (5)$$

where  $\chi = \chi(x, y, z, t, p)$ , and (3) is therefore integrable.

Here  $p$  plays the role of a variable spectral parameter, cf. e.g. [6, 7, 11, 15, 16, 17], just as in the (2+1)-dimensional case, which is arrived at when the dependence on  $z$  in  $\mathbf{u}$  and  $\chi$  is dropped.

Note that (5) implies the equation

$$(X_f(\chi))_t = (X_g(\chi))_y. \quad (6)$$

Substituting into (6) a formal expansion of  $\chi$  in  $p$  yields, upon subsequent splitting in  $p$ , an infinite hierarchy of (nonlocal) conservation laws for the associated integrable system (3), although some of those could be trivial, cf. e.g. [2, 5]. Using a formal expansion of  $\chi$  in  $p$  should also enable one, in analogy with e.g. [16, 18], to find, at least in some cases, infinite hierarchies of commuting flows for the system in question.

Smooth solutions of (5) form not just a vector space but a  $C^\infty$ -ring (see e.g. [19] for details on those), as an arbitrary smooth function of such solutions is again a solution. This is different from the (2+1)-dimensional case (cf. above), where we have Hamiltonian vector fields instead of the contact ones, so the Poisson bracket of smooth solutions is also a solution, and thus smooth solutions form [17] a Poisson  $C^\infty$ -algebra.

As an aside, remark that an example of a (3+1)-dimensional integrable system possessing a nonisospectral Lax pair that apparently cannot be transformed into the form (5) is provided by the Dunajski equation [20]; for the examples of nonisospectral Lax pairs in dimension higher than four see e.g. [16].

In closing reiterate that the existence of a linear Lax pair (5) along with (1) is of paramount importance as it definitely establishes integrability of the systems (3) associated with (1). In particular, there is a number of powerful methods for constructing plethora of exact solutions for these systems using (5): the inverse scattering transform and the dressing method in their various incarnations, see e.g. [11, 14, 17, 18, 21] and references therein, the twistor theory methods, see e.g. [7, 22], and some other techniques, cf. e.g. [23, 24, 25].

### 4. MORE ON INTEGRABLE SYSTEMS RELATED TO CONTACT LAX PAIRS

Upon having established integrability of systems (3) arising from (1) let us turn to their structure.

First of all, systems (3) are first-order quasi-linear, i.e., *dispersionless*, or *hydrodynamic-type*, systems, cf. e.g. [7, 11, 12, 17, 26, 27, 28, 29]. This is not too surprising, as the majority of hitherto known examples of (3+1)-dimensional integrable systems, including the celebrated (anti)self-dual Yang–Mills equations and the (anti)self-dual Einstein equations, also can be written as dispersionless systems, cf. e.g. [2, 7, 22].

On a related note, while the existence of linear Lax pairs (5) provides a definitive proof of integrability for the associated systems (3), checking whether the latter also pass an alternative integrability test specific to dispersionless systems, the existence of sufficiently many hydrodynamic reductions, see e.g. [28, 29] for details, could be of interest too. However, this problem is entirely beyond the scope of the present paper.

<sup>1</sup>More precisely, of  $S$ -integrability, as opposed to the so-called  $C$ -integrability, see [4].

Let us also mention that (1) can be written in a more symmetric parametric form,

$$\psi_x = \psi_z \Phi(\mathbf{u}, \zeta), \quad \psi_y = \psi_z \Theta(\mathbf{u}, \zeta), \quad \psi_t = \psi_z \Omega(\mathbf{u}, \zeta), \quad (7)$$

which could be more convenient, for instance, for the classification of associated integrable systems (3), cf. e.g. [12, 28] in the (2+1)-dimensional case. Expressing  $\zeta = \zeta(x, y, z, t)$  from the first equation of (7) and substituting the result into the remaining two equations gets us back to (1), to which we shall stick below.

In analogy with the (2+1)-dimensional case, cf. e.g. [11, 12, 29], it is clear that the class of (3+1)-dimensional integrable systems (3) arising from (1) is indeed fairly broad because of the freedom in choosing  $f$  and  $g$ .

Moreover, the very shape of (3) is actually not as restrictive as it may appear. For instance, we can rewrite in the form (3) a quasi-linear *second-order* equation

$$B_1 u_{xx} + B_2 u_{xy} + B_3 u_{xz} + B_4 u_{xt} + B_5 u_{yy} + B_6 u_{yz} + B_7 u_{yt} + B_8 u_{zz} + B_9 u_{zt} + B_{10} u_{tt} = 0, \quad (8)$$

where  $B_i = B_i(u_x, u_y, u_z, u_t)$ , if we put  $\mathbf{u} = (u_x, u_y, u_z, u_t)^T$ , and include the compatibility conditions like  $(u_x)_y = (u_y)_x$  into our system. Thus, we can look for integrable equations (8) with contact Lax pairs

$$\psi_y = \psi_z f(u_x, u_y, u_z, u_t, \psi_x/\psi_z), \quad \psi_t = \psi_z g(u_x, u_y, u_z, u_t, \psi_x/\psi_z). \quad (9)$$

It remains to be seen whether the integrability of these equations is related to the geometry of their linearizations in spirit of [30] (cf. also [31]).

## 5. EXAMPLES

**Example 1.** A simple example of an integrable system (3) possessing a contact Lax pair (1) arises if we put  $\mathbf{u} = (u, v, w, q)^T$ ,  $f = p^2 + wp + u$ , and  $g = p^3 + 2wp^2 + qp + v$ . Then (1) reads

$$\psi_y = \psi_x^2/\psi_z + w\psi_x + u\psi_z, \quad \psi_t = \psi_x^3/\psi_z^2 + 2w\psi_x^2/\psi_z + q\psi_x + v\psi_z,$$

and equating to zero the coefficients at the powers of  $p$  in (2) yields the system

$$\begin{aligned} u_t - vu_z - qu_x + uv_z + wv_x - v_y &= 0, \\ 2u_z + w_x + 2ww_z - q_z &= 0, \\ 2q_x - 3u_x - 2w_y + 2wu_z - v_z - 2ww_x + 2uw_z &= 0, \\ w_t - q_y + 2v_x - 4wu_x + wq_x - qw_x - vw_z + uq_z &= 0 \end{aligned} \quad (10)$$

that can be solved with respect to  $u_z, v_z, w_z, q_z$ , so it is actually an evolution system in disguise.

If  $u, v, w$  and  $q$  are independent of  $z$ , we can put  $w = 0$  and  $q = 3u/2$  in (10), and then it boils down to

$$4v_x = 3u_y, \quad 2u_t = 3uw_x + 2v_y. \quad (11)$$

Eliminating  $v$  yields an equation

$$(4u_t - 6uw_x)_x - 3u_{yy} = 0,$$

which, up to an obvious rescaling of  $y$  and  $u$ , is nothing but the celebrated dispersionless Kadomtsev–Petviashvili (dKP) equation, see e.g. [11, 27], also known as the ((2+1)-dimensional) Khokhlov–Zabolotskaya equation [32] or, when written in the potential form, as the Lin–Reissner–Tsien equation [33]. Thus, (10) provides a (3+1)-dimensional integrable generalization of the dKP equation.

It is readily checked that system (10) is not weakly nonlinear (or, in other terminology, not linearly degenerate, see e.g. [29, 34] for further details on this notion). We conjecture that all (3+1)-dimensional integrable systems possessing contact Lax pairs (1) share this property, i.e., they are *not* weakly nonlinear.

Example 1 can be easily generalized by letting  $f$  and  $g$  be polynomials of arbitrary degrees:

**Example 2.** Let  $m, n \in \mathbb{N}$ ,  $m > n$ ,  $N = m + n + 1$ ,  $\mathbf{u} = (u_0, \dots, u_n, v_0, \dots, v_{m-1})^T$ , and

$$f = p^{n+1} + \sum_{i=0}^n u_i p^i, \quad g = p^{m+1} + \frac{m}{n} u_n p^m + \sum_{j=0}^{m-1} v_j p^j.$$

Note that these  $f$  and  $g$  involve almost disjoint sets of dependent variables  $u_i$  and  $v_j$ :  $g$  involves  $v_j$ ,  $j = 0, \dots, m-1$ , and  $u_n$ , and  $f$  just  $u_i$ ,  $i = 0, \dots, n$ . This can be seen as a minimal generalization of the case, which in (2+1) dimensions was considered in [35], when  $f$  and  $g$  involve totally disjoint sets of dependent variables.

The associated contact Lax pair (1) reads

$$\psi_y = \psi_z \left( \left( \frac{\psi_x}{\psi_z} \right)^{n+1} + \sum_{i=0}^n u_i \left( \frac{\psi_x}{\psi_z} \right)^i \right), \quad \psi_t = \psi_z \left( \left( \frac{\psi_x}{\psi_z} \right)^{m+1} + \frac{m}{n} u_n \left( \frac{\psi_x}{\psi_z} \right)^m + \sum_{j=0}^{m-1} v_j \left( \frac{\psi_x}{\psi_z} \right)^j \right).$$

Equating to zero the coefficients at the powers of  $p$  in (2) yields a quadratic system of the form

$$\begin{aligned} & (u_k)_t - (v_k)_y + m(u_{k-m-1})_z - n(v_{k-n-1})_z + (n+1)(v_{k-n})_x - (m+1)(u_{k-m})_x \\ & + \sum_{i=\max(0, k-m)}^{\min(n, k)} ((k-i-1)v_{k-i}(u_i)_z - (i-1)u_i(v_{k-i})_z) - \sum_{i=\max(0, k+1-m)}^{\min(n, k+1)} ((k+1-i)v_{k+1-i}(u_i)_x - iu_i(v_{k+1-i})_x) = 0. \end{aligned} \quad (12)$$

Here  $k = 0, \dots, n+m$ , and for the ease of writing it is assumed that  $u_i = 0$  for  $i > n$  and  $i < 0$ ,  $v_j = 0$  for  $j > m$  and  $j < 0$ , and  $v_m = (m/n)u_n$ .

The number of equations in (12),  $n+m+1$ , is equal to the number of dependent variables  $u_i, v_j$ . Moreover, it is readily seen that (12) can be solved for the  $z$ -derivatives  $(u_i)_z$  and  $(v_j)_z$  for all  $i$  and  $j$ , i.e., (12) is an evolution system in disguise just like (10).

As  $m$  and  $n$  are arbitrary natural numbers such that  $m > n$ , (12) provides an example of a (3+1)-dimensional integrable system with an arbitrarily large finite number of components. In the simplest special case  $m = 2$  and  $n = 1$  we recover, modulo the notation, system (10) from Example 1.

Examples of integrable systems (3) whose contact Lax pairs (1) involve the functions  $f$  and  $g$  that are rational in  $p$ , or perhaps have an even more complicated form, cf. e.g. [12, 28], will be presented elsewhere.

## 6. REDUCTIONS TO (2+1)-DIMENSIONAL INTEGRABLE SYSTEMS

There are two substantially different ways to reduce (3), (1) and (2) to (2+1)-dimensional case: to drop the dependence of  $\mathbf{u}$  on  $y$  vs. on  $z$ . Let us treat them one by one.

If we drop the dependence on  $y$  in (2), we obtain a (nonlinear) Lax representation

$$f_t = \{g, f\}_L \quad (13)$$

which gives rise to integrable (2+1)-dimensional dispersionless systems of the form

$$A_0(\mathbf{u})\mathbf{u}_t + A_1(\mathbf{u})\mathbf{u}_x + A_3(\mathbf{u})\mathbf{u}_z = 0. \quad (14)$$

It should be possible, at least in some cases, to recover the original zero-curvature-type equation (2) and the associated (3+1)-dimensional integrable systems (3) through the central extension procedure, cf. e.g. [6, 36, 37].

On a related note, if we have two different Lax flows, (13) and  $f_y = \{h, f\}_L$ , then their compatibility condition is a weak form of the zero-curvature-type equation, namely,

$$\{h_t - g_y + \{h, g\}_L, f\}_L = 0. \quad (15)$$

Hence, if we have  $h_t - g_y + \{h, g\}_L = 0$ , which is nothing but (2) with  $f$  replaced by  $h$ , then (15) holds as well. This means that at least some of integrable systems (3) admitting the contact Lax pair (1) can arise as zero-curvature conditions for the commutativity of the Lax flows of the type (13).

Dropping dependence on  $t$  instead of  $y$  in (2) and (3) obviously leads to a similar picture with  $t \leftrightarrow y$ .

It remains to be seen whether new examples of integrable (2+1)-dimensional systems (14) could be obtained if we start directly from (13) with some generic  $f$  and  $g$  rather than just drop the dependence on  $y$  in (2).

On the other hand, if we assume that  $\mathbf{u}$  is independent of  $z$  and  $\psi$  satisfies  $\psi_z = 1$ , then the contact Lax pair (1) boils down to the well-studied (see e.g. [11, 12, 17, 28]) nonlinear Lax-type representation

$$\psi_y = f(\mathbf{u}, \psi_x), \quad \psi_t = g(\mathbf{u}, \psi_x), \quad (16)$$

which yields (2+1)-dimensional dispersionless integrable systems of the form

$$A_0(\mathbf{u})\mathbf{u}_t + A_1(\mathbf{u})\mathbf{u}_x + A_2(\mathbf{u})\mathbf{u}_y = 0. \quad (17)$$

Moreover, the nonisospectral linear Lax pair (5) also survives the reduction in question, if we assume that  $\chi$  is independent of  $z$ . Then (5) becomes nothing but the (2+1)-dimensional linear nonisospectral Lax pair written in terms of Hamiltonian (rather than contact) vector fields, a well-studied object, see e.g. [7, 17].

We can also consider yet another reduction of (1) and (3). Namely, put  $\psi_x = a = \text{const}$  and assume that  $\mathbf{u}$  is independent of  $x$ . Then (1) becomes

$$\psi_y = \psi_z f(\mathbf{u}, a/\psi_z), \quad \psi_t = \psi_z g(\mathbf{u}, a/\psi_z).$$

However, putting  $\tilde{f}(\mathbf{u}, \psi_z) = \psi_z f(\mathbf{u}, a/\psi_z)$  and  $\tilde{g}(\mathbf{u}, \psi_z) = \psi_z g(\mathbf{u}, a/\psi_z)$  immediately gets us back to the class of systems (17) with nonlinear Lax pairs (16), just with  $x$  replaced by  $z$ . Thus, the reduction in question does not appear to bring anything substantially new into the picture.

Note that for a nonlinear Lax pair (16) associated with a given (2+1)-dimensional integrable system (17), the contact Lax pair (1) with *exactly the same* functions  $f$  and  $g$  in general will yield a highly overdetermined, and possibly not even genuinely (3+1)-dimensional, system.

This is the case e.g. for the dKP equation in the form (11). Indeed, the nonlinear Lax pair (16) for (11) is

$$\psi_y = \psi_x^2 + u, \quad \psi_t = \psi_x^3 + \frac{3}{2}u\psi_x + v.$$

If we promote it to the form (1) while keeping the associated functions  $f$  and  $g$  unchanged, we obtain

$$\psi_y = \frac{\psi_x^2}{\psi_z} + u\psi_z, \quad \psi_t = \frac{\psi_x^3}{\psi_z^2} + \frac{3}{2}u\psi_x + v\psi_z,$$

but the compatibility condition  $(\psi_y)_t = (\psi_t)_y$  then yields an overdetermined system that consists of (11) and the equations  $u_z = 0$ ,  $v_z = 0$ , so we again get (11) itself instead of a (3+1)-dimensional generalization thereof.

## 7. OUTLOOK

We have presented above a broad new class of (3+1)-dimensional integrable dispersionless systems (3) associated with contact Lax pairs (1) and linear nonisospectral Lax pairs (5). This shows *inter alia* that integrable systems in (3+1) dimensions are significantly less exceptional than it appeared before, cf. e.g. [2, 5, 29].

Availability of linear nonisospectral Lax pairs for the systems under study makes it possible to solve the latter using powerful techniques like the inverse scattering transform or the dressing method, as discussed in Section 3. A related open problem is to find Hirota bilinear forms, cf. e.g. [38, 39, 40] and references therein, for integrable systems (3) and (8) associated with contact Lax pairs and find out which kinds of solutions for these systems could be constructed using the said forms.

It would also be interesting to explore the structure of symmetries, conservation laws, Hamiltonian structures, recursion operators, coverings, etc., for the systems in question, and to compare these objects with their counterparts in lower dimensions, cf. e.g. [2, 3, 6, 7, 26], [41]–[48], and references therein. This leads to another open problem: to find integrable systems (3) associated with (1) that can be written as systems of conservation laws, cf. e.g. [34, 49] for the latter.

On the other hand, some (2+1)-dimensional integrable dispersionless systems are known to be intimately related to the remarkable structures like infinite-dimensional Frobenius manifolds [50] or integrable hydrodynamic chains [51]. Finding new examples of such structures associated to integrable systems with contact Lax pairs is yet another challenge.

We conclude by listing three more important open problems related to (3+1)-dimensional integrable systems (3) and (8) possessing contact Lax pairs:

1. How, and under which conditions, can one construct dispersive deformations of these systems, i.e., integrable dispersive (3+1)-dimensional systems whose dispersionless limits yield the systems in question, cf. e.g. [52, 53, 54] and references therein for (2+1) and (1+1) dimensions?
2. Is it possible to quantize the systems under study, at least those possessing the Lagrangians, while preserving their integrability, in spirit of [8, 55, 56] and references therein, i.e., to construct (3+1)-dimensional exactly solvable quantum field theories that in the classical limit boil down to the systems in question?
3. How to discretize the systems under study and their contact Lax pairs while preserving the integrability, cf. e.g. [57] and references therein?

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